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APPROXIMATE REPRESENTATION

By W. E. BYERLY

1. Introduction. THAT the Least Square criterion employed in this paper could be made to give the received coefficients in a Fourier or in a Zonal Harmonic development appears to have been first noticed by Plarr who communicated his discovery through Bertrand to the *Comptes Rendus* in 1857.*

Later Toepler† and Gram‡ who were apparently unacquainted with Plarr's paper used practically the same method as Plarr; and these methods have been employed to some extent by recent writers on Integral Equations.§

So far as I have been able to discover these writers have usually confined their attention to the one-dimensional case, although some of them have mentioned that the methods they employed could be extended to the case of two-dimensional and three-dimensional problems.

That the methods here discussed furnish a very convenient and powerful tool in dealing with problems in Mathematical Physics which require the use of the so-called "Harmonic Analysis" is not, however, generally known, and I have found them so useful in my own work that it seemed to me worth while to communicate them to my fellow workers.

2. The Mean Square Criterion. Suppose that $f(x)$ and $F(x)$ are two given functions and that we wish to regard the second as an approximate representation of the first over a specified range of values for x , i.e., $x_0 < x < x_1$; is there a convenient criterion by which we can judge the excellence of the approximation?

When such a representation is used it is ordinarily desirable that the absolute value of the error $E \equiv f(x) - F(x)$, which of course is a function of x , should be small for all values of x considered. Of course in that case $E^2 \equiv [f(x) - F(x)]^2$ will be small and $[E^2]$, the mean value of E^2 over the range in question, will be small.

* *Comptes Rendus*, vol. 44, p. 985.

† *Wiener Anzeiger*, vol. 13, (1876), p. 205.

‡ *Crelle*, vol. 94, (1883), p. 41.

§ M. Bôcher, *An Introduction to the Study of Integral Equations*, pp. 52-60.

This $[E^2]$, the Mean Square of the Error, that is, the mean value of the square of the error $f(x) - F(x)$, over the range in question, we shall use as the measure of the excellence of our approximation. The smaller $[E^2]$, the better we shall call our representation.

Of course, since the mean of a set of positive quantities cannot be zero unless every one of the quantities is separately zero, our representation is *exact* if $[E^2] = 0$, and in that case $f(x)$ and $F(x)$ are equal for all values of x between x_0 and x_1 .

It is easily seen that

$$[E^2] \equiv \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} [f(x) - F(x)]^2 dx. \quad [I]$$

3. Determination of the "Best" Coefficients in an Approximate Representation of Specified Form. Let $f(x)$ and $\phi_1(x)$ be given functions* of x and let us try to determine a constant coefficient a_1 so that $a_1 \phi_1(x)$ shall *represent* $f(x)$ over the range from x_0 to x_1 as well as possible.

The problem is a very simple one in *maxima and minima*.

$$[E^2] \equiv \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} [f(x) - a_1 \phi_1(x)]^2 dx,$$

regarded as a function of a_1 , is to be made as small as possible. Write

$$\frac{d[E^2]}{da_1} \equiv -\frac{2}{x_1 - x_0} \int_{x_0}^{x_1} [f(x) - a_1 \phi_1(x)] \phi_1(x) dx = 0;$$

we get
$$a_1 = \frac{C_1}{A_1}, \quad [II]$$

where
$$A_1 = \int_{x_0}^{x_1} [\phi_1(x)]^2 dx$$

and
$$C_1 = \int_{x_0}^{x_1} f(x) \phi_1(x) dx;$$

then
$$[E^2] = \frac{1}{x_1 - x_0} \left[\int_{x_0}^{x_1} [f(x)]^2 dx - 2a_1 C_1 + a_1^2 A_1 \right].$$

* Throughout this paper we shall suppose that the functions employed are of a "respectability" above suspicion; finite, continuous, single-valued, integrable, and differentiable over the range or throughout the region considered. It is true that in many cases some of these limitations may be dispensed with, "but that is another story."

But from [II]

$$2a_1C_1 = 2a_1^2A_1.$$

Hence

$$[E^2] = \frac{1}{x_1 - x_0} \left[\int_{x_0}^{x_1} [f(x)]^2 dx - a_1^2 A_1 \right]. \quad [\text{III}]$$

4. Having found a_1 let us now determine a_2 so that the addition of the term $a_2 \phi_2(x)$, $\phi_2(x)$ being another given function, shall *improve* our representation as much as possible. To do this we have only to determine a_2 so that $a_2 \phi_2(x)$ shall represent $f(x) - a_1 \phi_1(x)$ as well as possible, and we get, by §3,

$$a_2 = \frac{\int_{x_0}^{x_1} f(x) \phi_2(x) dx - a_1 \int_{x_0}^{x_1} \phi_1(x) \phi_2(x) dx}{\int_{x_0}^{x_1} [\phi_2(x)]^2 dx},$$

or, if

$$A_1 = \int_{x_0}^{x_1} [\phi_1(x)]^2 dx, \quad A_2 = \int_{x_0}^{x_1} [\phi_2(x)]^2 dx, \quad C_1 = \int_{x_0}^{x_1} f(x) \phi_1(x) dx,$$

$$C_2 = \int_{x_0}^{x_1} f(x) \phi_2(x) dx, \text{ and } B_{1,2} = \int_{x_0}^{x_1} \phi_1(x) \phi_2(x) dx,$$

then

$$a_2 = \frac{C_2 - a_1 B_{1,2}}{A_2}; \quad [\text{IV}]$$

and

$$[E^2] = \frac{1}{x_1 - x_0} \left[\int_{x_0}^{x_1} [f(x)]^2 dx - a_1^2 A_1 - a_2^2 A_2 \right]. \quad [\text{V}]$$

5. Let us go back and determine the coefficients a_1 and a_2 *ab initio* so that $a_1 \phi_1(x) + a_2 \phi_2(x)$ shall represent $f(x)$ as well as possible over the range x_0 to x_1 .

Here we have a problem in *maxima* and *minima* of a function of two independent variables.

$$[E^2] \equiv \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} [f(x) - a_1 \phi_1(x) - a_2 \phi_2(x)]^2 dx;$$

write

$$\frac{\partial [E^2]}{\partial a_1} \equiv \frac{-2}{x_1 - x_0} \int_{x_0}^{x_1} [f(x) - a_1 \phi_1(x) - a_2 \phi_2(x)] \phi_1(x) dx = 0,$$

$$\frac{\partial [E^2]}{\partial a_2} \equiv \frac{-2}{x_1 - x_0} \int_{x_0}^{x_1} [f(x) - a_1 \phi_1(x) - a_2 \phi_2(x)] \phi_2(x) dx = 0.$$

To get a_1 and a_2 we have to solve the linear equations

$$\begin{aligned} A_1 a_1 + B_{1,2} a_2 &= C_1, \\ B_{1,2} a_1 + A_2 a_2 &= C_2, \end{aligned} \quad [\text{VI}]$$

where the notation is the same as in §4.

$$[E^2] = \frac{1}{x_1 - x_0} \left[\int_{x_0}^{x_1} [f(x)]^2 dx + a_1^2 A_1 + a_2^2 A_2 - 2a_1 C_1 - 2a_2 C_2 + 2a_1 a_2 B_{1,2} \right].$$

$$\text{From } [\text{VI}] \quad 2a_1 C_1 = 2a_1^2 A_1 + 2a_1 a_2 B_{1,2},$$

$$2a_2 C_2 = 2a_2^2 A_2 + 2a_1 a_2 B_{1,2}.$$

Hence

$$[E^2] = \frac{1}{x_1 - x_0} \left[\int_{x_0}^{x_1} [f(x)]^2 dx - a_1^2 A_1 - a_2^2 A_2 - 2a_1 a_2 B_{1,2} \right]. \quad [\text{VII}]$$

6. Example 1. (a) To determine a_1 so that $a_1 x$ shall be the best approximate representation of $\sin x$ from $x = 0$ to $x = \pi$.

Here

$$C_1 = \int_0^\pi x \sin x dx = \pi, \quad A_1 = \int_0^\pi x^2 dx = \frac{\pi^3}{3};$$

$$a_1 = \frac{C_1}{A_1} = \frac{3}{\pi^2} = 0.304;$$

and $\sin x = 0.304x$ approximately if $0 < x < \pi$.

As our test of the excellence of our approximation we have, since

$$\int_0^\pi \sin^2 x dx = \frac{\pi}{2},$$

$$[E^2] = \frac{1}{\pi} \left[\frac{\pi}{2} - \frac{9}{\pi^4} \frac{\pi^3}{3} \right] = \frac{1}{2} - \frac{3}{\pi^2} = 0.196.$$

It is interesting to note that should we take for $\sin x$ the first term of the familiar power series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

we should have over the same range

$$[E^2] = \frac{\pi^2}{3} - \frac{3}{2} = 1.789.$$

(b) To determine a_2 so that the addition of the term a_2x^3 shall improve our representation as much as possible.

Here A_1 and C_1 have the same values as in (a).

$$A_2 = \frac{\pi^7}{7}, \quad C_2 = \pi^3 - 6\pi, \quad \text{and } B_{1,2} = \frac{\pi^5}{5};$$

$$a_2 = \frac{14}{\pi^4} \left[\frac{1}{5} - \frac{3}{\pi^2} \right] = -0.015;$$

and for

$$\sin x = 0.304x - 0.015x^3$$

$$[E^2] = \frac{1}{\pi} \left[\frac{\pi}{2} - \frac{\pi^3}{3}a_1^2 - \frac{\pi^7}{7}a_2^2 \right] = 0.0307.$$

(c) To determine a_1 and a_2 so that $a_1x + a_2x^3$ shall represent $\sin x$ as well as possible over the range $0 < x < \pi$. Here A_1 , A_2 , C_1 , C_2 , and $B_{1,2}$ have the same values as in (b). We have

$$\frac{\pi^3}{3}a_1 + \frac{\pi^5}{5}a_2 = \pi, \quad \frac{\pi^5}{5}a_1 + \frac{\pi^7}{7}a_2 = \pi^3 - 6\pi.$$

Hence

$$a_1 = \frac{15}{2\pi^2} \left(\frac{21}{\pi^2} - 1 \right) = 0.8576,$$

$$a_2 = \frac{35}{2\pi^4} \left(1 - \frac{15}{\pi^2} \right) = -0.09348,$$

and

$$\sin x = 0.8576x - 0.09348x^3$$

approximately if $0 < x < \pi$. This last approximation is excellent since

$$[E^2] = \frac{1}{\pi} \left[\frac{\pi}{2} - \frac{\pi^3}{3}a_1^2 - \frac{\pi^7}{7}a_2^2 - 2\frac{\pi^5}{5}a_1a_2 \right] = 0.005.$$

Should we use as our approximation

$$\sin x = x - \frac{x^3}{3!}$$

we should have

$$[E^2] = 0.4008$$

over the same range.

7. Example 2. (a) To determine a_1 so that $a_1 \sin x$ shall represent x over the range from 0 to π as well as possible.

Here

$$A_1 = \int_0^\pi \sin^2 x \, dx = \frac{\pi}{2}, \quad C_1 = \int_0^\pi x \sin x \, dx = \pi,$$

$$a_1 = \frac{C_1}{A_1} = 2,$$

and

$$x = 2 \sin x$$

approximately ($0 < x < \pi$).

As our test of the excellence of the approximation we have

$$\left[E^2 \right] = \frac{1}{\pi} \left[\frac{\pi^3}{3} - 2\pi \right] = \frac{\pi^2}{3} - 2 = 1.29.$$

(b) To determine a_2 so that $a_2 \sin 3x$ shall improve the value just obtained as much as possible.

Here

$$A_2 = \int_0^\pi \sin^2 3x \, dx = \frac{\pi}{2},$$

$$C_2 = \int_0^\pi x \sin 3x \, dx = \frac{\pi}{3},$$

$$B_{1,2} = \int_0^\pi \sin x \sin 3x \, dx = 0,$$

so that

$$a_2 = \frac{2}{3};$$

here

$$\left[E^2 \right] = \frac{1}{\pi} \left[\frac{\pi^3}{3} - 2\pi - \frac{2}{9} \pi \right] = \frac{\pi^2}{3} - 2 - \frac{2}{9} = 1.07.$$

(c) To determine a_1 and a_2 so that $a_1 \sin x + a_2 \sin 3x$ shall represent x as well as possible over the range $0 < x < \pi$. We have

$$\frac{\pi}{2} a_1 = \pi, \quad \frac{\pi}{2} a_2 = \frac{\pi}{3},$$

so that

$$a_1 = 2, \quad \text{and} \quad a_2 = \frac{2}{3};$$

$$\left[E^2 \right] = \frac{1}{\pi} \left[\frac{\pi^3}{3} - 2\pi - \frac{2}{9} \pi \right] = 1.07.$$

8. Example 3. To determine a_1 and a_2 so that $a_1 \sin x + a_2 \sin^3 x$ shall represent $\sin 3x$ as well as possible over the range $0 < x < \frac{\pi}{2}$.

$$A_1 = \int_0^{\pi/2} \sin^2 x \, dx = \frac{\pi}{4},$$

$$A_2 = \int_0^{\pi/2} \sin^6 x \, dx = \frac{5\pi}{32},$$

$$B_{1,2} = \int_0^{\pi/2} \sin x \sin^3 x \, dx = \frac{3\pi}{16},$$

$$C_1 = \int_0^{\pi/2} \sin 3x \sin x \, dx = 0,$$

$$C_2 = \int_0^{\pi/2} \sin 3x \sin^3 x \, dx = -\frac{\pi}{16}.$$

$$\frac{\pi}{4} a_1 + \frac{3\pi}{16} a_2 = 0, \quad \frac{3\pi}{16} a_1 + \frac{5\pi}{32} a_2 = -\frac{\pi}{16}.$$

Hence

$$a_1 = 3,$$

$$a_2 = -4,$$

$$[E^2] = \frac{2}{\pi} \left[\frac{\pi}{4} - \frac{9\pi}{4} - 16 \cdot \frac{5\pi}{32} + 24 \cdot \frac{3\pi}{16} \right] = 0,$$

and the representation is exact, and

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

for all values of x between 0 and $\frac{\pi}{2}$.

9. The method of §5 is easily generalized. Let it be required to determine $a_1, a_2, \dots, a_n$ so that

$$\sum_{k=1}^{k=n} a_k \phi_k(x)$$

shall represent $f(x)$ over the range $x_0 < x < x_1$ as well as possible.

We find that the coefficients can be obtained by solving the n linear equations

$$\sum_{k=1}^{k=l-1} B_{k,l} a_k + A_l a_l + \sum_{k=l+1}^{k=n} B_{k,l} a_k = C_l, \quad (l = 1, 2, 3, \dots, n.) \quad [\text{VIII}]$$

$$[E^2] = \frac{1}{x_1 - x_0} \left\{ \int_{x_0}^{x_1} [f(x)]^2 dx - \sum_{k=1}^{k=n} A_k a_k^2 - 2 \sum_{k=1}^{k=n-1} \sum_{l=k+1}^{l=n} B_{k,l} a_k a_l \right\}, \quad [\text{IX}]$$

where

$$A_k = \int_{x_0}^{x_1} [\phi_k(x)]^2 dx,$$

$$B_{k,l} = \int_{x_0}^{x_1} \phi_k(x) \phi_l(x) dx,$$

and

$$C_k = \int_{x_0}^{x_1} f(x) \phi_k(x) dx.$$

If we desire to correct our representation as much as possible by the addition of another term $a_{n+1} \phi_{n+1}(x)$,

$$a_{n+1} = \frac{C_{n+1} - \sum_{k=1}^{k=n} a_k B_{k,n+1}}{A_{n+1}}, \quad [\text{X}]$$

and

$$[E_1^2] = \frac{1}{x_1 - x_0} \left\{ \int_{x_0}^{x_1} [f(x)]^2 dx - \sum_{k=1}^{k=n+1} A_k a_k^2 - 2 \sum_{k=1}^{k=n-1} \sum_{l=k+1}^{l=n} B_{k,l} a_k a_l \right\},$$

or

$$[E^2] = [E_1^2] - A_{n+1} a_{n+1}^2. \quad [\text{XI}]$$

10. Orthogonal Functions. A pair of functions the integral of whose product taken over the range $x_0 < x < x_1$ is zero we shall call orthogonal over the range.

That is if

$$\int_{x_0}^{x_1} \phi_m(x) \phi_n(x) dx = 0,$$

$\phi_m(x)$ and $\phi_n(x)$ are orthogonal over the range $x_0 < x < x_1$. A set of functions such that every pair is orthogonal over our range we shall call a *set of orthogonal functions* over the range.

If in §9 the functions $\phi_1(x)$, $\phi_2(x)$, $\dots$ $\phi_{n+1}(x)$ are *orthogonal* our results simplify surprisingly.

Since

$$B_{k,l} = \int_{x_0}^{x_1} \phi_k(x) \phi_l(x) dx = 0$$

[VIII] gives at once

$$A_k = \frac{C_k}{A_k}; \quad [\text{XII}]$$

[IX] gives

$$[E^2] = \frac{1}{x_1 - x_0} \left[\int_{x_0}^{x_1} [f(x)]^2 dx - \sum_{k=1}^{k=n} A_k a_k^2 \right]; \quad [\text{XIII}]$$

and [X] gives

$$a_{n+1} = \frac{C_{n+1}}{A_{n+1}}, \quad [\text{XIV}]$$

while [XI],

$$[E_1^2] = [E^2] - A_{n+1} a_{n+1}^2;$$

remains unchanged.

We see then that if we are trying to represent $f(x)$ as well as possible by the aid of a set of functions which are *orthogonal* over the range considered the coefficient a_k of a term $a_k \phi_k(x)$ which would be the best one if we used that term alone is the best one if that term is used in combination with any others of the set, and is the best one if that term is to be added as a correction to terms of the set whose best coefficients had been previously determined. Moreover since $[E^2]$ is necessarily positive no matter how many terms are used in the representation it is clear from formula (XIII) that the addition of any term with its appropriate coefficient* if the term can form an orthogonal pair with every term already present diminishes the Mean Square of the Error and improves the representation.

11. Corollaries. A few easy corollaries are :

(a) If $f(x)$ is capable of being exactly expressed in terms of the assumed functions for all values of x between x_0 and x_1 formula [VIII] §9 applied to the range $x_0 < x < x_1$ or to any portion of that range will give the correct coefficients. If it has not been foreseen that exact expression is possible formula [IX] will establish the fact by showing that $[E^2] = 0$. (Compare §8, Ex. 3).

(b) If it is known that for all values of x in a given range $f(x)$ can be expressed as a series of terms of the assumed form, and all the functions

* It must be kept in mind, however, that the appropriate coefficient C_k/A_k may happen to be zero, in which case of course the improvement is null.

$\phi_k(x)$ employed are *orthogonal* over the range, formula [XII] §10 will give the correct coefficients of the series.

(c) No development of $f(x)$ in a series whose terms are *orthogonal* over the range over which the development is to hold good can be correct unless the set of forms $\phi_k(x)$ employed is a *complete orthogonal* set; that is unless all forms are present which might be inserted without preventing the whole set employed from being *orthogonal*.*

(d) Since in computing the value of a function by the aid of a series into which it has been developed one can use only a limited number of terms, the check given by the value of $[E^2]$ is as valuable in practical applications in Physics when a valid series is employed for $f(x)$ as when an approximate representation is used.

(e) If in all that has gone before we replace *function* by *point-function*, *range* by *region* (one-, two-, or three-dimensional), and *integral over the range* by *integral throughout the region*, none of our formulas nor of our results will be changed in any essential particular.

12. Trigonometric Series. Since sines and cosines of whole multiples of x form a set of functions *orthogonal* over the range $-\pi < x < \pi$ the coefficients of the terms of Fourier type that best represent $f(x)$ over that range are given by formula [XII] §10.

But these are the familiar Fourier Coefficients.

Let $f(x) = \frac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + \dots + a_1 \sin x + a_2 \sin 2x + \dots$

Then
$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx,$$

and
$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx.$$

$$[E^2] = \frac{1}{2\pi} \left\{ \int_{-\pi}^{\pi} [f(x)]^2 \, dx - \pi \sum (a_m^2 + b_m^2) \right\}.$$

Of course if $f(x)$ is developable in a Fourier's Series our coefficients a_m and b_m are the correct coefficients (v. corollary (b) §11).

13. Let us obtain and test a few Trigonometric expressions for x over the range $0 < x < \pi$.

* Apparent exceptions to this corollary are due to the fact that the appropriate coefficient C_k/A_k for a term of proposed form may happen to be zero.

Here, since when m and n are integers $\int_0^\pi \sin mx \sin nx \, dx = 0$, and $\int_0^\pi \cos mx \cos nx \, dx = 0$, while $\int_0^\pi \sin mx \cos nx \, dx$ is not generally zero, we can deal conveniently with a sine expression

$$x = \sum a_m \sin mx,$$

or a cosine expression

$$x = \sum b_m \cos mx.$$

These are

$$x = 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots \right], \quad (1)$$

$$x = \frac{\pi}{2} - \frac{4}{\pi} \left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]. \quad (2)$$

For (1), $[E^2] = \frac{1}{\pi} \left[\frac{\pi^3}{3} - 2\pi \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right) \right] = 0$,

since $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

(v. Chrystal's Algebra vol. II, page 243), and as is well known (1) is a valid development of x over the range $0 < x < \pi$.

For (2)

$$[E^2] = \frac{1}{\pi} \left[\frac{\pi^3}{3} - \frac{\pi^3}{4} - \frac{16}{\pi^2} \frac{\pi^2}{2} \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right) \right] = 0,$$

since $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$

(v. Chrystal's Algebra, vol. I, page 243), whence

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96};$$

and as is well known (2) is a valid development of x .

$$\int_0^\pi \sin mx \cos nx \, dx = 0$$

if $m + n$ is even, that is if m and n are both odd or both even. Sines and cosines of odd multiples of x form, then, a set of functions *orthogonal* over the range $0 < x < \pi$, as do sines and cosines of even multiples of x . We can get

$$x = 2 \left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right) - \frac{4}{\pi} \left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right), \quad (3)$$

$$\text{and} \quad x = \frac{\pi}{2} - 2 \left(\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + \dots \right). \quad (4).$$

For (3),

$$[E^2] = \frac{1}{\pi} \left[\frac{\pi^3}{3} - 2\pi \left(\frac{1}{1^2} + \frac{1}{3^2} + \dots \right) - \frac{8}{\pi} \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right) \right] = 0,$$

$$\text{since} \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

For (4)

$$[E^2] = \frac{1}{\pi} \left[\frac{\pi^3}{3} - \frac{\pi^3}{4} - 2\pi \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots \right) \right] = 0,$$

and (3) and (4) are valid developments of x from $x = 0$ to $x = \pi$, which are not usually given in the texts.

14. Spherical Harmonics. Our method gives the familiar coefficients in the case of the other *Harmonic* developments.

(a) *Zonal Harmonics.* For instance since

$$\int_{-1}^1 P_m(x) P_n(x) dx = 0$$

if m and n are integers, the Zonal Harmonics $P_m(x)$ form a set of functions *orthogonal* over the range $-1 < x < 1$. If we wish to express $f(x)$ as

$$\sum a_m P_m(x),$$

$$a_m = \frac{\int_{-1}^1 f(x) P_m(x) dx}{\int_{-1}^1 [P_m(x)]^2 dx} = \frac{2m+1}{2} \int_{-1}^1 f(x) P_m(x) dx,$$

and

$$[E^2] = \frac{1}{2} \left[\int_{-1}^1 [f(x)]^2 dx - \sum \frac{2}{2m+1} a_m^2 \right].$$

The Zonal Harmonics $P_m(\cos \theta)$ are *orthogonal* over the surface of the unit sphere. (Byerly's *Fourier's Series* Art. 92).

If we wish to express $F(\theta)$ in terms of Zonal Harmonics and to use the unit sphere as our region we have

$$C_k = \int_0^{2\pi} \int_0^\pi F(\theta) P_k(\cos \theta) \sin \theta d\theta d\phi = 2\pi \int_0^\pi F(\theta) P_k(\cos \theta) \sin \theta d\theta.$$

$$A_k = \int_0^{2\pi} \int_0^\pi [P_k(\cos \theta)]^2 \sin \theta d\theta d\phi = 2\pi \left[\frac{2}{2k+1} \right],$$

and

$$a_k = \frac{C_k}{A_k},$$

the familiar value. Here

$$[E^2] = \frac{1}{4\pi} \left\{ 2\pi \int_0^\pi [F(\theta)]^2 \sin \theta d\theta - 4\pi \sum \frac{a_k^2}{2k+1} \right\}.$$

(b) *Tesseral Harmonics.* The Tesseral Harmonics of both types, i.e., $\cos n\phi P_m^n(\mu)$ and $\sin n\phi P_m^n(\mu)$, are a set of functions *orthogonal* over the unit sphere, unless in a pair of the same type the same subscript and the same index occur. (Byerly's *Fourier's Series*, Art. 105, and Art. 105, Ex. 2).

Hence if

$$a_{n,m} = \frac{C_{n,m}}{A_{n,m}},$$

where

$$C_{n,m} = \int_0^{2\pi} \int_0^\pi f(\theta, \phi) \cos n\phi P_m^n(\mu) \sin \theta d\theta d\phi,$$

$$A_{n,m} = \int_0^{2\pi} \int_0^\pi \cos^2 n\phi [P_m^n(\mu)]^2 \sin \theta d\theta d\phi = \frac{2\pi}{2m+1} \frac{(m+n)!}{(m-n)!},$$

and

$$b_{n,m} = \frac{C'_{n,m}}{A'_{n,m}},$$

where

$$C'_{n,m} = \int_0^{2\pi} \int_0^\pi f(\theta, \phi) \sin n\phi P_m^n(\mu) \sin \theta d\theta d\phi, \quad A'_{n,m} = A_{n,m},$$

and

$$a_{0,m} = \frac{C_{0,m}}{A_{0,m}},$$

where

$$C_{0,m} = \int_0^{2\pi} \int_0^\pi f(\theta, \phi) P_m(\mu) \sin \theta d\theta d\phi, \quad A_{0,m} = \frac{4\pi}{2n+1},$$

then

$$f(\theta, \phi) = \sum_{m=0}^{m=\infty} \left[a_{0,m} P_m(\mu) + \sum_{n=1}^{n=m} (a_{n,m} \cos n\phi + b_{n,m} \sin n\phi) P_m^n(\mu) \right],$$

(v. Byerly's *Fourier's Series*, Art. 107).

Here

$$\begin{aligned} [E^2] = \frac{1}{4\pi} & \left\{ \int_0^{2\pi} \int_0^\pi [f(\theta, \phi)]^2 \sin \theta d\theta d\phi \right. \\ & \left. - \sum_{m=0}^{m=\infty} \left[\frac{4\pi}{2m+1} a_{0,m}^2 + \sum_{n=1}^{n=m} \frac{2\pi}{2m+1} \frac{(m+n)!}{(m-n)!} (a_{n,m}^2 + b_{n,m}^2) \right] \right\}. \end{aligned}$$

15. Miscellaneous Examples. The examples which follow will give some notion of the convenience and power of the method of this paper in dealing with problems that require the use of the *Harmonic Analysis*.

In each of the first three examples a development of unity of importance in problems in Mathematical Physics is obtained and verified either directly or, as in Example 3, incidentally in the solution of an actual Physical Problem.

The developments in Example 1 and Example 2 are familiar, but are usually obtained by other methods, that in Example 3 I believe is new.

In Example 4 an approximate representation of unity in Bessel's Functions which are not *orthogonal* is considered.

In all the examples I refer freely to my *Fourier's Series* and *Spherical Harmonics*. (Boston, Ginn & Co.)

16. Example 1. To express 1 as

$$\sum \sum a_{m,n} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.*$$

Here our functions $\phi_{m,n}(x, y)$ are easily seen to be *orthogonal* over the rectangle bounded by the axes and the lines $x = a$, $y = b$.

* v. *Fourier's Series*, pp. 127, 128.

$$C_{m,n} = \int_0^a \int_0^b \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{b} dy dx = 0,$$

unless m and n are both odd, in which case

$$C_{m,n} = \frac{4ab}{mn\pi^2}.$$

$$A_{m,n} = \int_0^a \int_0^b \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} dy dx = \frac{ab}{4}.$$

$$\text{Hence } 1 = \frac{16}{\pi^2} \left[\sum_{m=0}^{m=\infty} \sum_{n=0}^{n=\infty} \frac{1}{(2m+1)(2n+1)} \sin \frac{(2m+1)\pi x}{a} \sin \frac{(2n+1)\pi y}{b} \right].$$

$$[E^2] = \frac{1}{ab} \left[ab - \frac{ab}{4} \frac{25b}{\pi^2} \sum_{m=0}^{m=\infty} \sum_{n=0}^{n=\infty} \frac{1}{(2m+1)^2(2n+1)^2} \right];$$

but the double series in the last term is easily seen to be the square of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots,$$

that is the square of $\pi^2/8$, and $[E^2] = 0$ and our expression is a correct development of 1 holding good for $0 < x < a$ and $0 < y < b$.

17. Example 2. The Bessel's Functions $J_0(\mu_k r)$, $J_0(\mu_l r)$ are orthogonal over the area of the circle whose centre is the origin and whose radius is a if and only if $\mu_k a$ and $\mu_l a$ are roots of the equation $J_0(x) = 0$, or of $J_1(x) = 0$, or of $xJ_1(x) - \lambda J_0(x) = 0$. (*Fourier's Series*, Art. 125.)

(a). Let us express 1 as $\sum a_k J_0(\mu_k r)$ if $\mu_k a$ is a root of $J_0(x) = 0$.

Here

$$a_k = \frac{C_k}{A_k},$$

where

$$C_k = \int_0^{2\pi} \int_0^a J_0(\mu_k r) r dr d\phi = 2\pi \int_0^a J_0(\mu_k r) r dr = \frac{2\pi a}{\mu_k} J_1(\mu_k a).$$

(*Fourier's Series*, p. 229 (6).)

$$A_k = \int_0^{2\pi} \int_0^a [J_0(\mu_k r)]^2 r dr d\phi = 2\pi \int_0^a [J_0(\mu_k r)]^2 r dr = \pi a^2 [J_1(\mu_k a)]^2,$$

and
$$a_k = \frac{2}{\mu_k a J_1(\mu_k a)};$$

and we get
$$1 = 2 \sum \frac{J_0(\mu_k r)}{\mu_k a J_1(\mu_k a)}. \quad (1)$$

Here
$$\begin{aligned} [E^2] &= \frac{1}{\pi a^2} \left[\pi a^2 - \sum \frac{4\pi a^2 [J_1(\mu_k a)]^2}{(\mu_k a)^2 [J_1(\mu_k a)]^2} \right] \\ &= \frac{1}{\pi a^2} \left[\pi a^2 - 4\pi a^2 \sum \frac{1}{(\mu_k a)^2} \right]. \end{aligned}$$

The equation

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \dots = 0$$

may be regarded as an equation in x^2 , and the sum of the reciprocals of its roots is the negative of the coefficient of its second term. Hence

$$\sum \frac{1}{(\mu_k a)^2} = \frac{1}{4}$$

and $[E^2] = 0$; and (1) is a correct development of 1 over the circle whose radius is a and therefore over the range $0 < r < a$.

(b) If $\mu_k a$ is a root of the equation $xJ_0(x) - \lambda J_1(x) = 0$, we have (v. *Fourier's Series*, p. 229)

$$C_k = \frac{2\pi a}{\mu_k} J_1(\mu_k a),$$

$$A_k = \frac{\pi(\lambda^2 + \mu_k^2 a^2) [J_0(\mu_k a)]^2}{\mu_k^2},$$

and
$$a_k = \frac{2\mu_k a}{(\lambda^2 + \mu_k^2 a^2) [J_0(\mu_k a)]^2} = \frac{2\lambda}{(\lambda^2 + \mu_k^2 a^2) J_0(\mu_k a)};$$

$$1 = 2\lambda \sum \frac{J_0(\mu_k r)}{(\lambda^2 + \mu_k^2 a^2) J_0(\mu_k a)}. \quad (2)$$

$$\begin{aligned} [E^2] &= \frac{1}{\pi a^2} \left[\pi a^2 - 4\lambda^2 \sum \frac{1}{(\lambda^2 + \mu_k^2 a^2)^2 [J_0(\mu_k a)]^2} \frac{\pi (\lambda^2 + \mu_k^2 a^2) [J_0(\mu_k a)]^2}{\mu_k^2} \right] \\ &= \frac{1}{\pi a^2} \left[\pi a^2 - 4\pi a^2 \sum \frac{\lambda^2}{(\lambda^2 + \mu_k^2 a^2) \mu_k^2 a^2} \right]. \end{aligned}$$

$$\frac{\lambda^2}{(\lambda^2 + \mu_k^2 a^2) \mu_k^2 a^2} = \frac{1}{(\mu_k a)^2} - \frac{1}{\lambda^2 + \mu_k^2 a^2}.$$

The equation

$$xJ_1(x) - \lambda J_0(x) = -\lambda + \left(\frac{1}{2} + \frac{\lambda}{4}\right)x^2 - \left(\frac{1}{2^4} + \frac{\lambda}{2^2 4^2}\right)x^4 + \dots = 0$$

may be regarded as an equation in x^2 .

The sum of the reciprocals of its roots is the negative of the coefficient of the second term divided by the first term or $(2 + \lambda)/(4\lambda)$; that is,

$$\sum \frac{1}{\mu_k^2 a^2} = \frac{2 + \lambda}{4\lambda}.$$

To get

$$\sum \frac{1}{\lambda^2 + \mu_k^2 a^2}$$

is not so easy.

We must first transform $F(x^2) = xJ_1(x) - \lambda J_0(x) = 0$ into an equation whose roots exceed those of $F(x^2) = 0$ by λ^2 .

The constant term will be $F'(-\lambda^2)$ and the coefficient of the term of first degree will be $F'(-\lambda^2)$. (v. Todhunter's *Theory of Equations*, §54.).

$$\text{Here} \quad F'(x^2) = \frac{1}{2x} [xJ_0(x) + \lambda J_1(x)] = \frac{1}{2} \left[J_0(x) + \frac{\lambda J_1(x)}{x} \right].$$

$$\sum \frac{1}{\lambda^2 + \mu_k^2 a^2} = -\frac{F'(-\lambda^2)}{F(-\lambda^2)} = -\frac{J_0(\lambda i) + \frac{\lambda J_1(\lambda i)}{\lambda i}}{2[\lambda i J_1(\lambda i) - \lambda J_0(\lambda i)]}$$

$$= -\frac{1}{2\lambda} \frac{J_0(\lambda i) - iJ_1(\lambda i)}{iJ_1(\lambda i) - J_0(\lambda i)} = \frac{1}{2\lambda}.$$

$$\sum \left[\frac{1}{\mu_k^2 a^2} - \frac{1}{\lambda^2 + \mu_k^2 a^2} \right] = \frac{2 + \lambda}{4\lambda} - \frac{1}{2\lambda} = \frac{1}{4}.$$

Hence $[E^2] = 0$ and (2) is a correct development of 1 over the circle whose radius is a , and therefore over the range $0 < r < a$.

18. Example 3. A homogeneous cylinder of radius c and altitude b is initially at the constant temperature unity throughout. Its entire surface is suddenly cooled to the temperature zero and kept there. Required the temperature of any point in the cylinder at any time after cooling has begun.

We have to satisfy the heat conduction equation

$$D_t u = a^2 [D_r^2 u + \frac{1}{r} D_r u + D_z^2 u] \quad (1)$$

subject to the conditions

$$u = 0 \text{ when } z = 0 \quad (2)$$

$$u = 0 \text{ when } z = b \quad (3)$$

$$u = 0 \text{ when } r = c \quad (4)$$

$$u = 1 \text{ when } t = 0 \quad (5)$$

It is not difficult to find and it is easy to verify the particular solution

$$u = e^{-(\nu^2 + \mu^2)t} J_0(\mu r) \sin \nu z,$$

where μ and ν are any constants. This value satisfies equation (1) and condition (2).

If $\nu = \frac{m\pi}{b}$ it satisfies (3). If $\mu = \mu_k$, when $\mu_k c$ is a root of $J_0(x) = 0$, it satisfies (4).

Let
$$u = \sum \sum a_{k,m} e^{-\left(\frac{m^2\pi^2}{b^2} + \mu_k^2\right)at} J_0(\mu_k r) \sin \frac{m\pi z}{b}.$$

This will be the required solution if

$$1 = \sum \sum a_{k,m} J_0(\mu_k r) \sin \frac{m\pi z}{b}$$

throughout the cylinder.

The functions $J_0(\mu_k r) \sin \frac{m\pi z}{b}$ are easily seen to be orthogonal throughout the cylinder.

$$a_{k,m} = \frac{C_{k,m}}{A_{k,m}},$$

where

$$\begin{aligned} C_{k,m} &= \int_0^b \int_0^{2\pi} \int_0^c J_0(\mu_k r) \sin \frac{m\pi z}{b} r \, dr \, d\phi \, dz \\ &= \frac{2cb}{m\mu_k} (1 - \cos m\pi) J_1(\mu_k c); \quad (\text{Four. Ser., page 229.}) \end{aligned}$$

$$A_{k,m} = \frac{\pi c^2 b}{2} [J_1(\mu_k c)]^2; \quad (\text{Four. Ser., page 229.})$$

$$a_{k,m} = \frac{4(1 - \cos m\pi)}{m\pi \mu_k c J_1(\mu_k c)};$$

and our expression for 1 is

$$1 = \frac{4}{\pi} \sum \sum \frac{1 - \cos m\pi z}{m\mu_k c J_1(\mu_k c)} J_0(\mu_k r) \sin \frac{m\pi z}{b}.$$

$$[E^2] = \frac{1}{\pi c^2 b} \left[\pi c^2 b - \sum \sum \frac{8b c^2 (1 - \cos m\pi)^2}{m^2 \pi \mu_k c^2} \right].$$

$$\sum_{m=1}^{m=\infty} \frac{(1 - \cos m\pi)^2}{m^2 \pi} = \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] = \frac{4}{\pi} \frac{\pi^2}{8} = \frac{\pi}{2}$$

$$\sum_{k=1}^{k=\infty} \frac{1}{(\mu_k c)^2} = \frac{1}{4} \quad (\text{v. Ex. 2 (a) §17}).$$

Hence $[E^2] = 0$, and our expression for 1 is exact and

$$u = \frac{8}{\pi} \sum_{m=0}^{m=\infty} \sum_{k=1}^{k=\infty} \frac{1}{(2m+1) \mu_k c J_1(\mu_k c)} J_0(\mu_k r) \sin \frac{(2m+1)\pi z}{b} e^{-\left(\frac{m^2 \pi^2}{b^2} + \mu_k^2\right) a^2 t}$$

rigorously.

19. Example 4. In the preceding examples the functions in terms of which we have had occasion to express unity have in every case been orthogonal over the range or throughout the region considered. As an example of a different sort suppose we wish to express 1 in terms of Bessel's Functions $J_0(\mu_k x)$ which are not orthogonal over the circle over which the expression is to be used.*

* For a case of this sort see a paper by B. O. Peirce, *Proc. Amer. Acad.*, vol. 43, No. 5, Sept. 1907, p. 175, pp. 180-184.

There is no serious difficulty in getting any specified number of terms by the formulas of §6 of this paper.

We have

$$\begin{aligned} C_l &= \int_0^\pi \int_0^a J_0(\mu_l r) r \, dr \, d\phi = \frac{2\pi a}{\mu_l} J_1(\mu_l a) ; \\ A_l &= \int_0^{2\pi} \int_0^a [J_0(\mu_l r)]^2 r \, dr \, d\phi = \pi a^2 \{ [J_0(\mu_l a)]^2 + [J_1(\mu_l a)]^2 \} ; \\ B_{k,l} &= \int_0^{2\pi} \int_0^a J_0(\mu_k r) J_0(\mu_l r) r \, dr \, d\phi \\ &= \frac{1}{\mu_k^2 - \mu_l^2} [\mu_k a J_0(\mu_l a) J_1(\mu_k a) - \mu_l a J_0(\mu_k a) J_1(\mu_l a)] . \end{aligned}$$

(v. *Fourier's Series*, §§125-126, pp. 228-229.)

For instance, if a single term $a_1 J_0(\mu_1 r)$ will suffice,

$$\begin{aligned} a_1 &= \frac{C_1}{A_1} = \frac{2J_1(\mu_1 a)}{\mu_1 a \left[\left(J_0(\mu_1 a) \right)^2 + \left(J_1(\mu_1 a) \right)^2 \right]} . \\ [E^2] &= 1 - \frac{4[J_1(\mu_1 a)]^2}{\mu_1^2 a^2 \left[\left(J_0(\mu_1 a) \right)^2 + \left(J_1(\mu_1 a) \right)^2 \right]} . \end{aligned}$$

If we need three terms we have

$$A_1 a_1 + B_{1,2} a_2 + B_{1,3} a_3 = C_1 ,$$

$$B_{1,2} a_1 + A_2 a_2 + B_{2,3} a_3 = C_2 ,$$

$$B_{1,3} a_1 + B_{2,3} a_2 + A_3 a_3 = C_3 ,$$

and

$$[E^2] = \frac{1}{\pi a^2} \left[\pi a^2 - A_1 a_1^2 - A_2 a_2^2 - A_3 a_3^2 - 2 \left(B_{1,2} a_1 a_2 + B_{2,3} a_2 a_3 + B_{1,3} a_1 a_3 \right) \right] .$$

20. Conclusion. It may be objected that the test of excellence used in this paper is a purely artificial one. It is, however, often a very useful one and for many purposes it is precisely what is wanted ; and it is in fact the test that a specified number of terms in any of the received Harmonic developments always fulfil.

The method given here like other rough and ready methods, for instance that of Lagrange (*Fourier's Series*, §§19-25, *et passim*), has nothing to do, except indirectly, with the question of development in series. Indeed if a development in series of the proposed form is possible the coefficients of a specified set of terms given by the method of Least Squares are not the coefficients of the corresponding terms of the series unless the functions happen to be orthogonal throughout the region considered. In other cases they give a closer approximation than would the actual terms of the series. (v. Ex. 1 (c) §6.)

The check on the accuracy of the representation in any concrete case given by the value of $[E^2]$ is useful even when terms of a true series development are employed and it is easily applied in any case since all the constants in $[E^2]$ must have been calculated already in getting the coefficients.

If the functions used in the representation of $f(x)$ are orthogonal throughout the region in question the labor of getting the coefficients is much lessened, since every one is obtained separately, and every additional one obtained adds to the accuracy of the result.

If the functions employed are not orthogonal throughout the region the coefficients obtained are for many purposes the best possible, and whether or not the representation is accurate enough for the matter in hand is at once shown by the value of $[E^2]$.

In the subjects which gave rise to the Harmonic Analysis, i.e., in the Motions of Elastic Strings and in the Conduction of Heat, most of the developments required are in terms of functions orthogonal over the range or throughout the region concerned, and the old theory worked well enough. In problems in Electricity and Magnetism, and in Hydromechanics similar developments where the functions are not orthogonal are often desired and to them the old theory gives no clue.

Although our method has nothing to do with development in series except indirectly, corollaries (b) and (c) §11 show that it may often be of great service in that connection, and they suggest some beautiful little problems for the workers in the science of Integral Equations.

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